Impedance Matching Equation: Developed Using Wheeler’s Methodology

IEEE Long Island Section Antennas & Propagation Society
Presentation
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Outline

1. Background Information
2. The Impedance Matching Equation
3. The Bode and Fano Impedance Matching Equations
4. Wheeler’s Single- and Double-Tuning Equations
5. Conversion of Wheeler’s Equations to the Original Impedance Matching Equation
6. Development of the final form for the Impedance Matching Equation
7. A note on Triple-Tuned Impedance Matching
1940s
Wheeler develops impedance matching principles
A Wheeler designed double-tuned impedance-matched IFF antenna played a critical role in WW II
Bode and Fano publish their work on impedance matching

1950
Wheeler publishes Report 418, a tutorial on impedance matching that features the reflection chart as a primary tool
For single- and double-tuned impedance matching, it presents three equations that quantify impedance-matching bandwidth limitations related to a specified maximum reflection magnitude
Based on the works of Bode and Fano, it quantifies the law of diminishing returns for impedance matching circuits beyond double tuning

1973
Wheeler’s three equations are converted to the original Impedance Matching Equation

2004
Using MATCAD to solve Fano’s equations, the final version of the Impedance Matching Equation was developed
Impedance-Matching Equation

\[ B_n(\Gamma) = \frac{1}{Q} b_n \sinh \left( \frac{1}{a_n} \ln \left( \frac{1}{\Gamma} \right) \right) + \frac{1 - b_n}{a_n} \ln \left( \frac{1}{\Gamma} \right) \]

Assumes Lumped-Element Circuits  Exact for \( n = 1, 2, \text{ and } \infty \)

\( QB_n \) Error < 0.1% for \( \Gamma > 0.10 \) (Max VSWR > 1.2)

\( B_n = \text{Maximum fractional impedance-matching bandwidth} \)
\( B_n = (f_H - f_L)/f_0 \)
\( f_0 = \text{Resonant frequency} = \sqrt{f_H f_L} \)
\( Q = \text{Antenna Q (Ratio of reactive power to radiated and dissipated power)} \)
\( \Gamma = \text{Maximum reflection magnitude within } B_n \)
\( n = \text{Number of tuned stages in the impedance matching circuit} \)

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Bode Impedance Matching Equation
(Hendrik W. Bode)

\[
B = \frac{1}{Q} \pi \ln \left( \frac{1}{\Gamma} \right) \\
Q = \frac{\omega_0 L}{R}
\]

B = Theoretical maximum fractional bandwidth for specified maximum reflection magnitude
Fano’s Impedance Matching Equations
(Robert M. Fano)

\[
\Gamma = \frac{\cosh(nb)}{\cosh(\alpha)} = \frac{\tanh(\alpha)}{\cosh(\alpha)} = \frac{\tanh(nb)}{\cosh(b)}
\]

\[
2 \sin \left( \frac{\pi}{2n} \right) = \frac{QB}{\sinh(a) - \sinh(b)}
\]

\[
QB_n(\Gamma)
\]

NOTE: The Impedance Matching Equation is a closed-form approximate solution for the Fano Impedance Matching Equations

n tuned stages
Alternate - series and parallel
All stages tuned to \( f_0 \)
n = 1 is the tuned antenna
The Bode-Fano Equation

Fano showed that in the limit case of $n = \infty$

$$B_\infty = \frac{1}{Q} \frac{\pi}{\ln\left(\frac{1}{\Gamma}\right)}$$
We Started in 1973 With Wheeler’s Three Equations for a Resonant Antenna

1950 Wheeler Lab Report 418

1. \( Q_B = \tan(\varphi_{EB}) \) \( \varphi_{EB} \) = Magnitude of impedance phase at edge-band frequencies

2. \( \Gamma_1 = \tan\left(\frac{\varphi_{EB}}{2}\right) \) (Optimum Single Tuning)

3. \( \Gamma_2 = \Gamma_1^2 \) (Optimum Double Tuning)
Wheeler’s First Equation

Wheeler’s Small Resonant Antenna
Lumped-Element RLC Circuit
Example: Small Electric Dipole
  Capacitor resonated with series L

\[
Z_{EB} = R + j \frac{1}{\omega_0 C} \left( \frac{f_H}{f_0} - \frac{f_0}{f_H} \right)
\]

\[
Z_{EB} = R \left( 1 + j \frac{1}{\omega_0 CR} \left( \frac{f_H}{f_0} - \frac{f_L}{f_0} \right) \right)
\]

\[
Z_{EB} = R(1 + jQB) = R \cdot \exp(j\varphi_{EB})
\]

\[
\tan(\varphi_{EB}) = QB
\]
Wheeler’s Optimum Single- and Double-Tuned Impedance Matching (Proof by Inspection)

**Single Tuning (Mid-Band Match)**
\[ \tan(\varphi_{EB}/2) = QB \]

**Optimum Single Tuning (Edge-Band Match)**
\[ \Gamma_1 = \tan(\varphi_{EB}/2) \]
Impedance transformation can not reduce \( \Gamma_1 \)

**Double Tuning**
\[ \Gamma_2 = \Gamma_1^2 \]
Impedance transformation and/or change in Q of second tuning stage can not reduce \( \Gamma_2 \)

\( \varphi_{EB} \) = Impedance phase at edge frequencies, \( f_H \) and \( f_L \)
Single Tuning: Derivation of \( |\Gamma_{EB}| = \tan\left(\frac{\phi_{EB}}{2}\right) \)

From Reflection Chart

\( R_0 = 1 \)

\( \Gamma_{EB} = \frac{e^{j\phi_{EB}} - 1}{e^{j\phi_{EB}} + 1} \)

\( \Gamma_{1} = |\Gamma_{EB}| = \sqrt{\cos^2(\phi_{EB}) - 2 \cos(\phi_{EB}) + 1 + \sin^2(\phi_{EB})} \)

\( \Gamma_{1} = \frac{\sqrt{1 - \cos(\phi_{EB})}}{\sqrt{1 + \cos(\phi_{EB})}} = \tan\left(\frac{\phi_{EB}}{2}\right) \)
Derivation of $\Gamma_2 = \Gamma_1^2$

Similar Triangles

\[
\frac{\Gamma_2}{\Gamma_1} = \frac{\Gamma_1}{1}
\]

$\Gamma_2 = \Gamma_1^2$
In 1973 we converted Wheeler’s three equations for a resonant antenna to a single equation

1. \( QB = \tan(\varphi_{EB}) \quad \varphi_{EB} = \text{Impedance phase at edge frequency} \)
2. \( \Gamma_1 = \tan(\varphi_{EB}/2) \)  (Single Tuning)
3. \( \Gamma_2 = \Gamma_1^2 \)  (Double Tuning)

\[
\text{Single Tuning:} \quad \tan(\varphi) = \frac{2 \tan(\varphi/2)}{1 - \tan^2(\varphi/2)} \quad QB_1 = \frac{2\Gamma_1}{1 - \Gamma_1^2}
\]

\[
\text{Double Tuning:} \quad QB_2 = \frac{2\sqrt{\Gamma_2}}{1 - \Gamma_2}
\]

Wheeler’s Equation:
Single tuning, \( n = 1 \)  \( B_n(\Gamma) = \frac{1}{Q} \frac{2\Gamma^n}{1 - \Gamma^n} \)
Double tuning, \( n = 2 \)
At this point we had an explicit expression that related $B$, $Q$, $\Gamma$, and $n$ for single- and double-tuned impedance matching.

We were aware of the Bode and Fano results.

Wheeler clearly defined the law of diminishing returns for added stages beyond double tuning.

One remaining question was: How much bandwidth increase can be achieved with triple tuning over that of double tuning?
Wheeler’s Equation: \( QB_n = \frac{2\Gamma^n}{1 - \Gamma^n} \)

\[
QB_1 = \frac{2\Gamma}{1 - \Gamma^2} = \frac{2}{\frac{1}{\Gamma} - \Gamma} = \frac{2}{e^{\ln\left(\frac{1}{\Gamma}\right)} - e^{-\ln\left(\frac{1}{\Gamma}\right)}} = \frac{1}{\sinh\left(\ln\left(\frac{1}{\Gamma}\right)\right)}
\]

\[
QB_2 = \frac{2\sqrt{\Gamma}}{1 - \Gamma} = \frac{2}{\frac{1}{\sqrt{\Gamma}} - \sqrt{\Gamma}} = \frac{2}{e^{\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)} - e^{-\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)}} = \frac{1}{\sinh\left(\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)\right)}
\]

\[
B_n = \frac{1}{Q} \frac{1}{\sinh\left(\frac{1}{a_n} \ln\left(\frac{1}{\Gamma}\right)\right)} \approx \frac{1}{Q} \frac{a_n}{\ln\left(\frac{1}{\Gamma}\right)} \quad \text{for } \Gamma > \frac{1}{3}
\]

\( a_1 = 1, \text{ and } a_2 = 2 \)
Bode - Fano Equation

\[ B_\infty = \frac{1}{Q} \frac{\pi}{\ln \left( \frac{1}{\Gamma} \right)} \quad a_\infty = \pi \]

For all \( n \) and \( \Gamma > 1/3 \):

Is \( B_n \approx \frac{1}{Q} \frac{a_n}{\ln \left( \frac{1}{\Gamma} \right)} \) ???

Knew that \( a_1 = 1 \), \( a_2 = 2 \), and \( a_\infty = \pi \)


\[
1 + \frac{1}{3} \frac{1}{3} \left( \frac{2}{3} \right)^2 + \frac{1}{7} \left( \frac{4}{3} \right)^2 + \ldots \ldots \infty \frac{\pi}{2}
\]

\[
1 + \frac{1}{3} + \frac{1}{3} \frac{1}{3} \left( \frac{2}{3} \right)^2 + \frac{1}{5} \left( \frac{2}{3} \right)^2 + \frac{1}{7} \left( \frac{4}{3} \right)^2 \quad + \ldots \ldots \infty = \pi
\]

\[
a_n = \sum_{k=1}^{n} s_k \quad a_1 = 1 \quad a_2 = 2 \quad a_3 = 2.333 \quad a_4 = 2.667 \quad a_5 = 2.756 \ldots \quad a_\infty = \pi
\]
1973 Impedance-Matching Equation
(Original Equation)

\[ B_n (\Gamma) \approx \frac{1}{Q} \frac{1}{\sinh\left(\frac{1}{a_n} \ln\left(\frac{1}{\Gamma}\right)\right)} \]

Exact for \( n = 1 \) and \( 2 \)
Approximate for \( \Gamma > 1/3 \), and \( n > 2 \)

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<th>( a_n )</th>
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For \( \Gamma = 1/3 \)
\[ \frac{B_2}{B_1} = 2.31 \text{ (131% Increase)} \]
\[ \frac{B_\infty}{B_2} = 1.65 \text{ (65% Increase)} \]
\[ \frac{B_3}{B_2} = 1.18 \text{ (18% Increase ?)} \]

Sent letter to Professor Fano asking for help in determining accuracy of \( a_n \)
1973 Fano’s Reply

\[
\frac{A_1^\infty}{\omega_c} = \frac{\sinh(a) - \sinh(b)}{\sin\left(\frac{\pi}{2n}\right)} \tag{36}
\]

\[
\Gamma = \frac{\cosh(nb)}{\cosh(na)} \tag{37}
\]

\[
\frac{\tanh(na)}{\cosh(a)} = \frac{\tanh(nb)}{\cosh(b)} \tag{38}
\]

\[
\ln \frac{1}{|\rho_1|_{\text{MAX.}}} = \frac{\pi}{QB} \tag{b)
\]

\[\begin{align*}
\frac{A_1^\infty}{\omega_c} &= \frac{2R}{L} \\
\frac{A_1^\infty}{\omega_c} &= \frac{2R}{\omega_0L} \frac{\omega_c}{\omega_0} \\
\frac{A_1^\infty}{\omega_c} &= \frac{2}{QB}
\end{align*}\]

Fig. 19. Tolerance of match for a low-pass ladder structure with n elements
2004 – Comparison of Fano and Original Matching Equation

\[ \frac{1}{\Gamma} = \frac{\ln \left( \frac{1}{\Gamma} \right)}{\pi} \]

\[ \frac{1}{QB_{\infty}} = \sinh \left( \frac{1}{\ln \left( \frac{1}{\Gamma} \right)} \right) \]

\[ \frac{1}{QB_{n}} = \sinh \left( \frac{1}{a_{n} \ln \left( \frac{1}{\Gamma} \right)} \right) \]

\[ \frac{1}{QB_{2}} = \sinh \left( \frac{1}{2 \ln \left( \frac{1}{\Gamma} \right)} \right) \]

\[ \frac{1}{QB_{1}} = \sinh \left( \ln \left( \frac{1}{\Gamma} \right) \right) \]

\[ \ln \frac{1}{\Gamma} \]

\[ \frac{1}{QB} \]

\[ \Gamma > \frac{1}{3} \]

Used MATHCAD to solve Fano’s equations.
2004 Impedance-Matching Equation

$$B_n(\Gamma) = \frac{1}{Q} \left( \frac{1}{b_n \sinh \left( \frac{1}{a_n \ln \left( \frac{1}{\Gamma} \right)} \right)} + \frac{1 - b_n}{a_n \ln \left( \frac{1}{\Gamma} \right)} \right)$$

$b_n$ coefficient provides blending of the “sinh” and “ln” functions

$$B_3/B_2 = 1.24 \ (24\% \ Increase)$$
Conclusion

• Wheeler’s development of the principles for double-tuned impedance matching was a major contribution. Although it was developed for lumped-element circuits it has a broader application
• One can see by inspection that his solutions were optimum
• We have developed the Impedance-Matching Equation, a closed form solution for the Fano Equations, which we hope will be helpful and useful to the community
• What impressed me the most in all of this work was the remarkable fact that Wheeler’s results, using the reflection chart, were identical to the results obtained by Fano using high-level network theory
Wheeler and Fano

**Wheeler (Reflection Chart)**

\[ B_n(\Gamma) = \frac{1}{Q} \frac{2\Gamma^n}{1 - \Gamma^n} \]

\[ B_1(\Gamma) = \frac{1}{Q} \frac{2}{\Gamma - \Gamma} \]

**Fano (Network Theory)**

\[ B_n(\Gamma) = \frac{1}{Q} \frac{2\sin\left(\frac{\pi}{2n}\right)}{\sinh(a) - \sinh(b)} \]

\[ \frac{\tanh(\text{na})}{\cosh(a)} = \frac{\tanh(\text{nb})}{\cosh(b)} \]

\[ \frac{\cosh(\text{nb})}{\cosh(\text{na})} = \Gamma \]

\[ B_1(\Gamma) = \frac{1}{Q} \frac{2}{\sinh(a) - \sinh(b)} \]

\[ \sinh(a) = \frac{1}{\Gamma} \quad \sinh(b) = \Gamma \]
Triple-Tuned Impedance Matching
Which circle, A or B, should be used to position the edge-band frequencies on the Max $\Gamma$ Circle

Circle A or Circle B

Max $\Gamma$ Circle
$\Gamma = 1/2$
VSWR = 3

$f_L$ and $f_H$

Double-Tuned Locus
Triple-Tuned Impedance Matching Cont’d

Edge-Band Frequencies on Horizontal Axis

Edge-Band Frequencies on Vertical Axis
Triple-Tuned Impedance Matching Cont’d
Triple-Tuned Monopole Antenna
On Infinite Ground Plane
Triple-Tuned Monopole Antenna (Continued)

Double Tuned

Triple Tuned

Mg=0.33
Ph=-136°
701 MHz

Mg=0.33
Ph=82°
315 MHz
Triple-Tuned Monopole Antenna (Continued)

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![Graph showing VSWR and $S_{11}$ with markers for 2.00 U at 315 MHz, 328 MHz, 612 MHz, and 703 MHz.](image)