Impedance-Matching Equation: Developed Using Wheeler’s Methodology

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Impedance-Matching Equation

\[ B_n(\Gamma) = \frac{1}{Q} \cdot \frac{1}{b_n \sinh\left(\frac{1}{a_n} \ln\left(\frac{1}{\Gamma}\right)\right) + \frac{1-b_n}{a_n} \ln\left(\frac{1}{\Gamma}\right)} \]

Exact for \( n = 1, 2, \) and \( \infty \)

\( QB_n \) Error < 0.1% for \( \Gamma > 0.10 \), and < 0.3% for \( \Gamma > 0.05 \)

\( B_n = \) Fractional impedance-matching bandwidth
\( B_n = (f_H - f_L)/f_0 \)
\( f_0 = \) Resonant frequency = \( \sqrt{f_H f_L} \)
\( Q = \) Antenna Q (Ratio of reactive power to radiated and dissipated power)
\( \Gamma = \) Maximum reflection magnitude within \( B_n \)
\( n = \) Number of tuned stages in the impedance matching circuit (Measure of the complexity of the circuit)

<table>
<thead>
<tr>
<th>n</th>
<th>a_n</th>
<th>b_n</th>
<th>n</th>
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Wheeler’s Optimum Single- and Double-Tuned Matching Proof by Inspection

Optimum Single Tuning (Edge-Band Match)
\[ \Gamma_1 = \tan(\varphi_{EB}/2) \]
Impedance transformation can not reduce \( \Gamma_1 \)

Optimum Double Tuning
\[ \Gamma_2 = \Gamma_1^2 \]
Impedance transformation and/or change in Q of second tuning stage can not reduce \( \Gamma_2 \)

\( \varphi_{EB} = \) Impedance phase at edge frequencies
Wheeler’s three equations (1940s) for a resonant antenna were converted to a single equation

1. \(QB = \tan(\varphi)\) \(\varphi = \) Impedance phase at edge frequency
2. \(\Gamma_1 = \tan(\varphi/2)\) (Single Tuning)
3. \(\Gamma_1 = \sqrt{\Gamma_2}\) (Double Tuning)

Single Tuning: \(\tan(\varphi) = \frac{2 \tan(\varphi/2)}{1 - \tan^2(\varphi/2)}\)
\(QB_1 = \frac{2\Gamma_1}{1 - \Gamma_1^2}\)

Double Tuning: \(QB_2 = \frac{2\sqrt{\Gamma_2}}{1 - \Gamma_2}\)

Wheeler’s Equation:
Single tuning, \(n = 1\) \(B_n = \frac{1}{Q} \frac{\Gamma_1^n}{1 - \Gamma_1^n}\)
Double tuning, \(n = 2\) \(B_n = \frac{1}{Q} \frac{2\Gamma_1^n}{1 - \Gamma_1^2}\)
Wheeler’s Equation: \[ QB_n = \frac{2\Gamma^n}{1 - \Gamma^n} \]

\[ QB_1 = \frac{2\Gamma}{1 - \Gamma^2} = \frac{2}{\Gamma - \Gamma} = \frac{2}{\Gamma} = \frac{1}{\Gamma} \approx e^{\ln\left(\frac{1}{\Gamma}\right)} - e^{-\ln\left(\frac{1}{\Gamma}\right)} = \frac{1}{\sinh\left(\ln\left(\frac{1}{\Gamma}\right)\right)} \]

\[ QB_2 = \frac{2\sqrt{\Gamma}}{1 - \Gamma} = \frac{1}{\sqrt{\Gamma} - \sqrt{\Gamma}} = \frac{2}{\sqrt{\Gamma}} \approx e^{\frac{1}{2} \ln\left(\frac{1}{\Gamma}\right)} - e^{-\frac{1}{2} \ln\left(\frac{1}{\Gamma}\right)} = \frac{1}{\sinh\left(\frac{1}{2} \ln\left(\frac{1}{\Gamma}\right)\right)} \]

\[ B_n = \frac{1}{Q} \frac{1}{\sinh\left(\frac{1}{a_n} \ln\left(\frac{1}{\Gamma}\right)\right)} \approx \frac{1}{Q} \frac{a_n}{\ln\left(\frac{1}{\Gamma}\right)} \text{ for } \Gamma > \frac{1}{3} \]

\[ a_1 = 1, \text{ and } a_2 = 2 \]
1973 Continued

Fano - Bode Equation

\[ B_\infty = \frac{1}{Q} \frac{\pi}{\ln\left(\frac{1}{\Gamma}\right)} \quad a_\infty = \pi \]

For all \( n \) and \( \Gamma > 1/3 \):

Is \( B_n \approx \frac{1}{Q} \frac{a_n}{\ln\left(\frac{1}{\Gamma}\right)} \) ???

Knew that \( a_1 = 1 \), \( a_2 = 2 \), and \( a_\infty = \pi \)


\[
1 + \frac{1}{3} + \frac{1}{5} \left(\frac{2}{3}\right)^2 + \frac{1}{7} \left(\frac{2}{3}\right)^2 + \ldots \ldots \infty = \frac{\pi}{2}
\]

\[
1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{3} + \frac{1}{5} \left(\frac{2}{3}\right)^2 + \frac{1}{7} \left(\frac{2}{3}\right)^2 + \frac{1}{7} \left(\frac{2}{3}\right)^2 + \ldots \ldots \infty = \pi
\]

\[ a_n = \sum_{k=1}^{n} s_k \quad a_1 = 1 \quad a_2 = 2 \quad a_3 = 2.333 \quad a_4 = 2.667 \quad a_5 = 2.756 \ldots \quad a_\infty = \pi \]
1973 Impedance-Matching Equation
(Original Equation)

\[ B_n (\Gamma) = \frac{1}{Q} \frac{1}{\sinh \left( \frac{1}{a_n} \ln \left( \frac{1}{\Gamma} \right) \right)} \]

Exact for \( n = 1 \) and \( 2 \)
Approximate for \( \Gamma > 1/3 \), and \( n > 2 \)

<table>
<thead>
<tr>
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<th>( a_n )</th>
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Sent letter to Professor Fano asking for help in determining accuracy of \( a_n \)
Fig. 19. Tolerance of match for a low-pass ladder structure with n elements
2004 – Comparison of Fano and Original Matching Equation

\[
\frac{1}{\text{QB}_\infty} = \frac{\ln \left( \frac{1}{\Gamma} \right)}{\pi}
\]

\[
\frac{1}{\text{QB}_n} = \sinh \left( \frac{1}{\ln \left( \frac{1}{\Gamma} \right)} \right)
\]

\[
\frac{1}{\text{QB}_1} = \sinh \left( \ln \left( \frac{1}{\Gamma} \right) \right)
\]

\[
\frac{1}{\text{QB}_2} = \sinh \left( \frac{1}{2} \ln \left( \frac{1}{\Gamma} \right) \right)
\]

\[
\frac{1}{\text{QB}_3} = \sinh \left( \frac{1}{3} \ln \left( \frac{1}{\Gamma} \right) \right)
\]

\[
\frac{1}{\text{QB}_\infty} = \sinh \left( \frac{1}{n} \ln \left( \frac{1}{\Gamma} \right) \right)
\]

\[
\Gamma > \frac{1}{3}
\]
2004 Impedance-Matching Equation

\[
B_n(\Gamma) = \frac{1}{Q} \frac{1}{b_n \sinh\left(\frac{1}{a_n \ln\left(\frac{1}{\Gamma}\right)}\right) + \frac{1-b_n}{a_n \ln\left(\frac{1}{\Gamma}\right)}}
\]

\(b_n\) coefficient provides blending of the “\(\sinh\)” and “\(\ln\)” functions
Conclusion

• Wheeler’s development of the principles for double-tuned impedance matching was a major contribution. Although it was developed for lumped-element circuits it has a broader application
• You can see by inspection that his solutions were optimum
• The Impedance-Matching Equation provides connectivity and a good perspective for the works of Wheeler and Fano. Although Wheeler described qualitatively the law of diminishing returns for multiple-tuned circuits beyond double tuning, Fano’s work quantified this tradeoff
• Wheeler once said: “You have to work hard to find the easy way”
• Wheeler’s simple geometrical development of optimum single- and double-tuned matching, using the reflection chart, was truly a work of art.
A Work Of Art

Harold A Wheeler

Fig. 8, WL Report 418, May 1950