Impedance-Matching Equation: Developed Using Wheeler's Methodology

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Impedance-Matching Equation

$$B_n(\Gamma) = \frac{1}{Q} \frac{1}{b_n \sinh\left(\frac{1}{a_n} \ln\left(\frac{1}{\Gamma}\right)\right) + \frac{1-b_n}{a_n} \ln\left(\frac{1}{\Gamma}\right)}$$

Exact for n = 1, 2, and ∞

QB_n Error < 0.1% for $\Gamma > 0.10$, and < 0.3% for $\Gamma > 0.05$

B_n = Fractional impedance-matching bandwidth

$B_n = (f_H - f_L)/f_0$

f_0 = Resonant frequency = $\sqrt{f_H f_L}$

Q = Antenna Q (Ratio of reactive power to radiated and dissipated power}

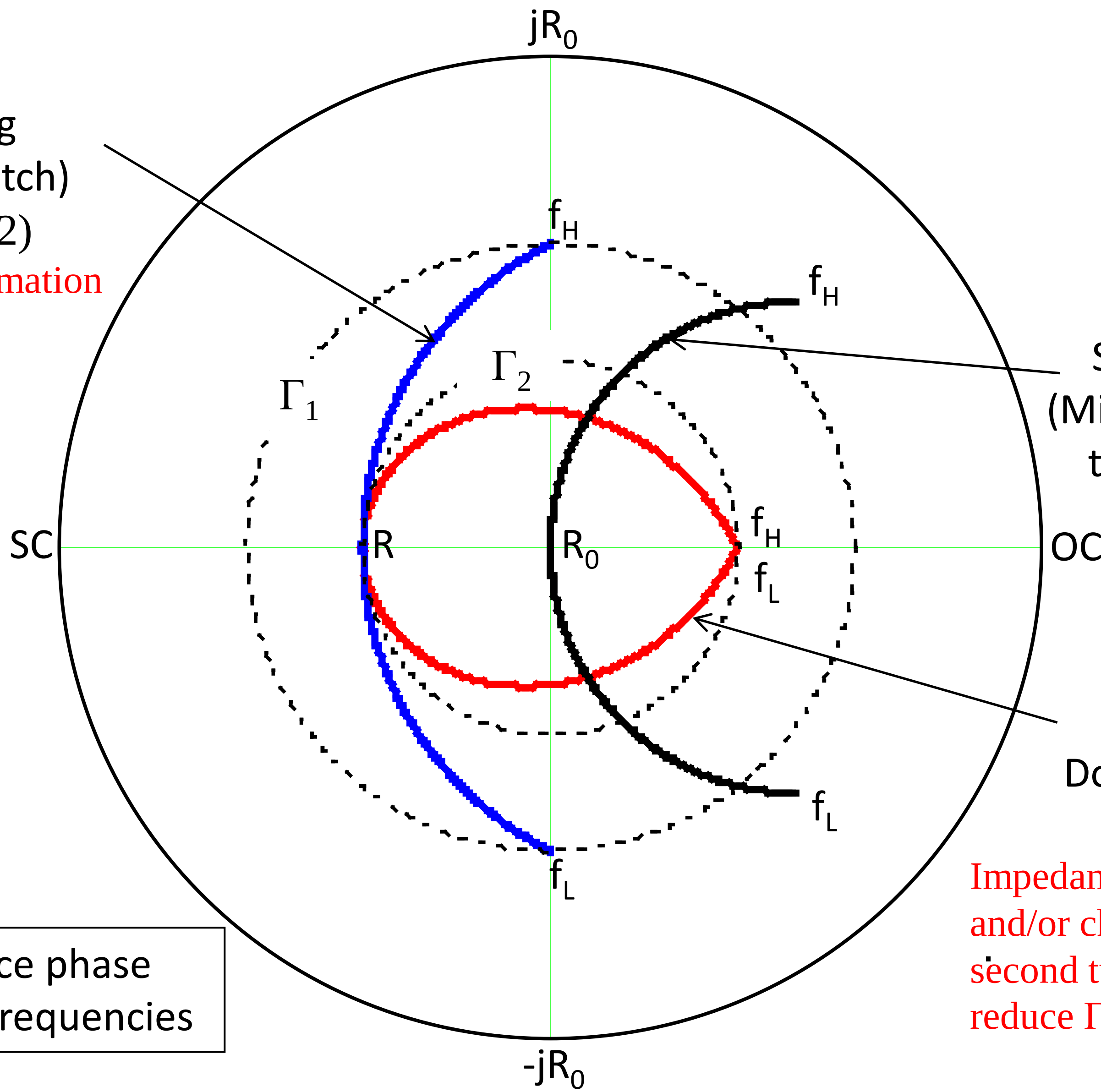
Γ = Maximum reflection magnitude within B_n

n = Number of tuned stages in the impedance matching circuit (Measure of the complexity of the circuit)

n	**a_n**	**b_n**		**n**	**a_n**	**b_n**
1	1	1		6	2.838	0.264
2	2	1		7	2.896	0.209
3	2.413	0.678		8	2.937	0.160
4	2.628	0.474				
5	2.755	0.347		∞	π	0

Wheeler's Optimum Single- and Double-Tuned Matching
Proof by Inspection

Optimum
Single Tuning
(Edge-Band Match)
$\Gamma_1 = \tan(\varphi_{EB}/2)$
Impedance transformation can not reduce Γ_1



Single Tuning
(Mid-Band Match)
$\tan(\varphi_{EB}) = QB$

Optimum
Double Tuning
$\Gamma_2 = \Gamma_1^2$
Impedance transformation and/or change in Q of second tuning stage can not reduce Γ_2

φ_{EB} = Impedance phase at edge frequencies

1973

Wheeler's three equations (1940s) for a resonant antenna were converted to a single equation

1. $QB = \tan(\varphi)$ φ = Impedance phase at edge frequency
2. $\Gamma_1 = \tan(\varphi/2)$ (Single Tuning)
3. $\Gamma_1 = \sqrt{\Gamma_2}$ (Double Tuning)

Single Tuning : $\tan(\varphi) = \dfrac{2\tan(\varphi/2)}{1-\tan^2(\varphi/2)}$ $QB_1 = \dfrac{2\Gamma_1}{1-\Gamma_1^2}$

Double Tuning : $QB_2 = \dfrac{2\sqrt{\Gamma_2}}{1-\Gamma_2}$

Wheeler's Equation:
Single tuning, n = 1
Double tuning, n = 2

$$B_n = \frac{1}{Q}\frac{2\Gamma^{\frac{1}{n}}}{1-\Gamma^{\frac{2}{n}}}$$

1973 Continued

Wheeler's Equation: $$QB_n = \frac{2\Gamma^{\frac{1}{n}}}{1-\Gamma^{\frac{2}{n}}}$$

⇩

$$QB_1 = \frac{2\Gamma}{1-\Gamma^2} = \frac{2}{\frac{1}{\Gamma}-\Gamma} = \frac{2}{e^{\ln\left(\frac{1}{\Gamma}\right)} - e^{-\ln\left(\frac{1}{\Gamma}\right)}} = \frac{1}{\sinh\left(\ln\left(\frac{1}{\Gamma}\right)\right)}$$

$$QB_2 = \frac{2\sqrt{\Gamma}}{1-\Gamma} = \frac{2}{\frac{1}{\sqrt{\Gamma}}-\sqrt{\Gamma}} = \frac{2}{e^{\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)} - e^{-\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)}} = \frac{1}{\sinh\left(\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)\right)}$$

⇩

$$B_n = \frac{1}{Q}\frac{1}{\sinh\left(\frac{1}{a_n}\ln\left(\frac{1}{\Gamma}\right)\right)} \approx \frac{1}{Q}\frac{a_n}{\ln\left(\frac{1}{\Gamma}\right)} \quad \text{for } \Gamma > \frac{1}{3}$$

$$a_1 = 1, \text{ and } a_2 = 2$$

1973 Continued

Fano - Bode Equation

$$B_\infty = \frac{1}{Q}\frac{\pi}{\ln\left(\frac{1}{\Gamma}\right)} \qquad a_\infty = \pi$$

For all n and $\Gamma > 1/3$:

$$\text{Is } B_n \approx \frac{1}{Q}\frac{a_n}{\ln\left(\frac{1}{\Gamma}\right)} ???$$

Knew that $a_1 = 1$, $a_2 = 2$, and $a_\infty = \pi$

Ref.: L.B.W. Jolley, "Summation of Series," Dover, New York, (410), p. 76, 1961

$$1+\frac{1}{3}+\frac{1}{5}\left(\frac{2}{3}\right)^2+\frac{1}{7}\left(\frac{2}{3}\frac{4}{5}\right)^2+\ldots\ldots\ldots\infty=\frac{\pi}{2}$$

$$1+1+\frac{1}{3}+\frac{1}{3}+\frac{1}{5}\left(\frac{2}{3}\right)^2+\frac{1}{5}\left(\frac{2}{3}\right)^2+\frac{1}{7}\left(\frac{2}{3}\frac{4}{5}\right)^2+\frac{1}{7}\left(\frac{2}{3}\frac{4}{5}\right)^2+\ldots\ldots\ldots\infty=\pi$$

$$a_n = \sum_{k=1}^{n} s_k \quad a_1 = 1 \quad a_2 = 2 \quad a_3 = 2.333 \quad a_4 = 2.667 \quad a_5 = 2.756\ldots \quad a_\infty = \pi$$

1973 Impedance-Matching Equation (Original Equation)

$$B_n(\Gamma) = \frac{1}{Q} \frac{1}{\sinh\left(\frac{1}{a_n} \ln\left(\frac{1}{\Gamma}\right)\right)}$$

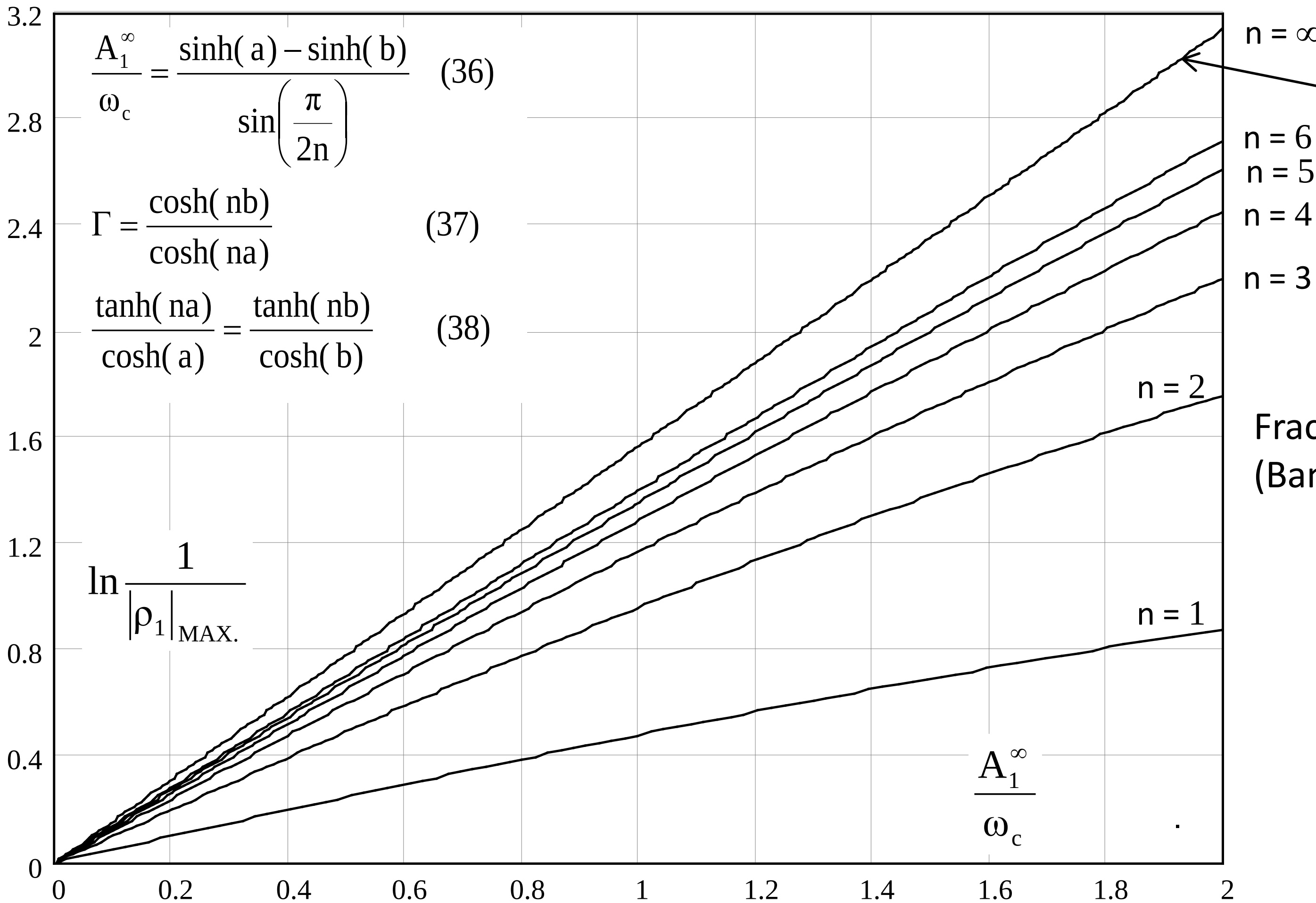
Exact for n = 1 and 2

Approximate for Γ > 1/3, and n > 2

n	**a_n**		**n**	**a_n**
1	1		6	2.84
2	2		7	2.89
3	2.33		8	2.93
4	2.67			
5	2.76		∞	π

Sent letter to Professor Fano asking for help in determining accuracy of a_n

1973 Fano's Reply



$$\ln\frac{1}{\Gamma} = \frac{\pi}{QB}$$

Fractional Bandwidth (Band-Pass) Conversion

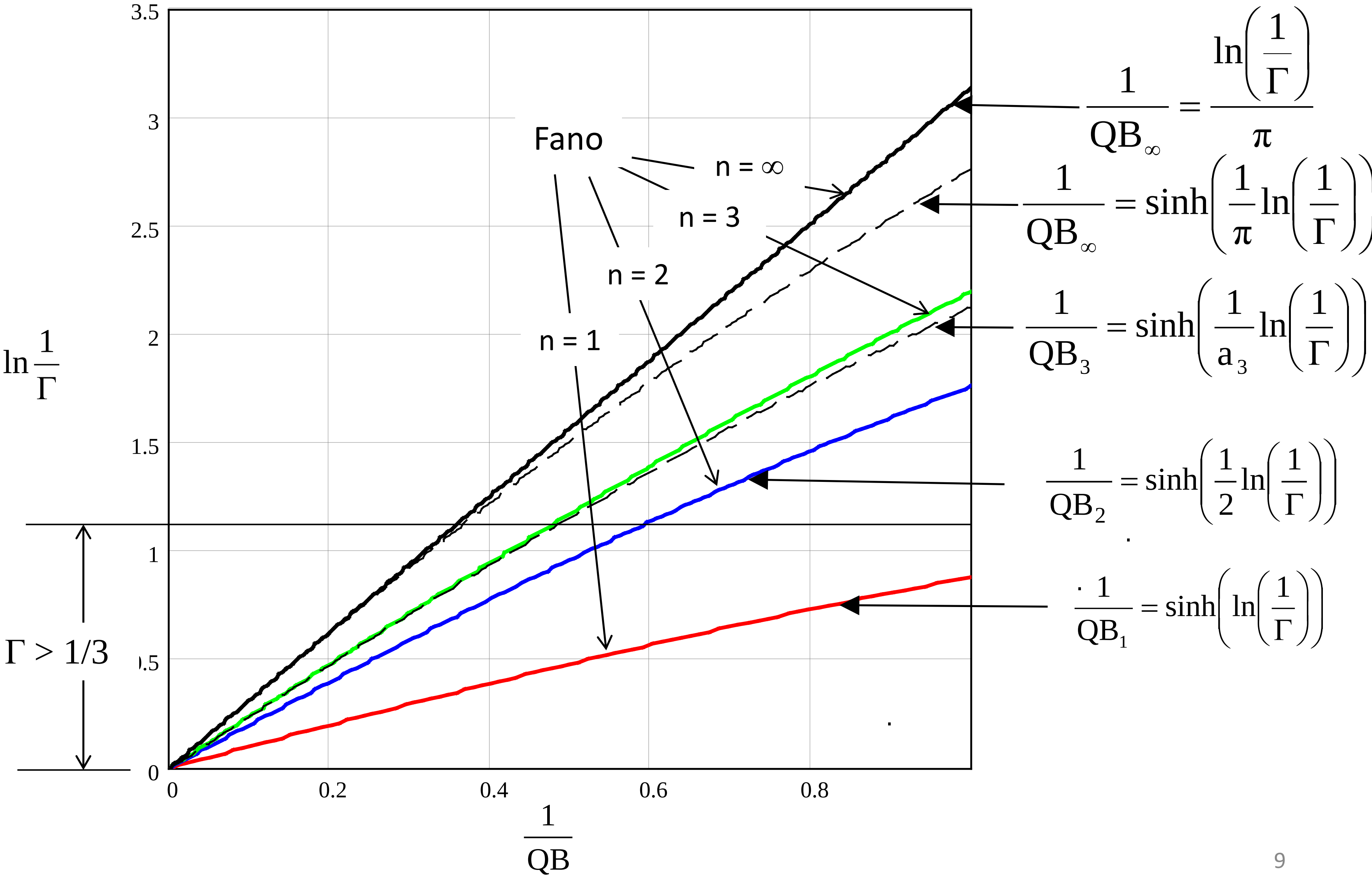
$$A_1^\infty = \frac{2R}{L}$$

$$\frac{A_1^\infty}{\omega_c} = \frac{2R}{\omega_0 L}\frac{\omega_0}{\omega_c}$$

$$\frac{A_1^\infty}{\omega_c} = \frac{2}{QB}$$

Fig. 19. Tolerance of match for a low-pass ladder structure with n elements

2004 – Comparison of Fano and Original Matching Equation



2004 Impedance-Matching Equation

$$B_n(\Gamma) = \frac{1}{Q} \frac{1}{b_n \sinh\left(\frac{1}{a_n}\ln\left(\frac{1}{\Gamma}\right)\right) + \frac{1-b_n}{a_n}\ln\left(\frac{1}{\Gamma}\right)}$$

b_n coefficient provides blending of the "sinh" and "ln" functions

Conclusion

- Wheeler’s development of the principles for double-tuned impedance matching was a major contribution. Although it was developed for lumped-element circuits it has a broader application
- You can see by inspection that his solutions were optimum
- The Impedance-Matching Equation provides connectivity and a good perspective for the works of Wheeler and Fano. Although Wheeler described qualitatively the law of diminishing returns for multiple-tuned circuits beyond double tuning, Fano’s work quantified this tradeoff
- Wheeler once said: “You have to work hard to find the easy way”
- Wheeler’s simple geometrical development of optimum single- and double-tuned matching, using the reflection chart, was truly a work of art.

A Work Of Art

Harold A Wheeler

Fig. 8, WL Report 418, May 1950